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Geometric Algebra and Singularities arising in Differential Line Geometry

Junki Tanaka (Kobo Co, LTD) and Toru Ohmoto (Hokkaido Univ.)
田中純貴（(株) 工房），大本　亨（北海道大学大学院理学研究院）

This is a digest report of [12, 13]; we give an elementary characterization of local diffeomorphic types of **singular ruled/developable surfaces** in $\mathbb{R}^3$ and their bifurcations by using **dual quaternions** and **$\mathcal{A}$-classification theory of map-germs**. Maps and manifolds are of class C^∞ throughout.

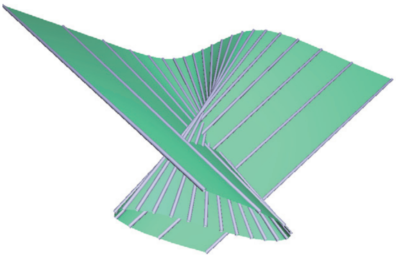

Fig. 1 Deforming Mond's H_2-singularity via a family of ruled surfaces.

1 Geometric Algebra

Geometric Algebra is a new look at Clifford algebras; It provides very neat tools for describing motions in Klein geometries in the context of a vast of applications to physics, mechanics and computer vision etc (cf. e.g. [11]). First we present a quick introduction.

1.1 Clifford algebra

The Clifford algebra $Cl(p,q,r)$ is the quotient of the non-commutative polynomial ring of n indeterminates $\mathbf{e}_1, \cdots, \mathbf{e}_n$ with real coefficients ($n = p+q+r$), i.e., the tensor algebra $\bigoplus_{r=0}^{\infty} V_1^{\otimes r}$ of $V_1 = \bigoplus_{i=1}^{n} \mathbb{R}\mathbf{e}_i$, via the two-sided ideal corresponding to relations

$$\mathbf{e}_i^2 = 1 \ \ (1 \le i \le p), \quad \mathbf{e}_{p+i}^2 = -1 \ \ (1 \le i \le q), \quad \mathbf{e}_{p+q+i}^2 = 0 \ \ (1 \le i \le r)$$
$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0 \ \ (i \ne j).$$

It is graded: $Cl(p,q,r) = \mathbb{R} \oplus V_1 \oplus \cdots \oplus V_n = Cl^+ \oplus Cl^-$ (een/odd parts), where we put $V_k = \bigoplus_{i_1<\cdots<i_k} \mathbb{R}\,\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k}$, called the space of k-**blades**.

Example 1.1

- $Cl(0,0,0) = \mathbb{R}$, $\quad Cl(0,1,0) = \mathbb{C} \quad (\mathbf{e}_1 = \sqrt{-1})$
- $Cl(0,0,1) = \mathbb{R}[\varepsilon]/\langle\varepsilon^2\rangle = \mathbb{R} \oplus \varepsilon\mathbb{R} =: \mathbb{D}$: **Dual numbers** $a + b\varepsilon$ $(\varepsilon^2 = 0)$
- $Cl(0,2,0)$=**Hamilton's quaternions** $(\mathbf{e}_1 = i, \mathbf{e}_2 = j, \mathbf{e}_3 = k)$

$$\mathbb{H} := \mathbb{R} \oplus \mathrm{Im}\,\mathbb{H} = \{q = a + bi + cj + dk = a + \boldsymbol{v}\}, \quad \boldsymbol{v} = (b, c, d)^T$$

- $Cl^+(0,3,1)$= **Dual quaternions**:

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{D} = \mathbb{H} \oplus \varepsilon\mathbb{H} = \{\, \check{q} = q_0 + \varepsilon q_1 \mid q_0, q_1 \in \mathbb{H} \,\}$$

	$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{D}$	$\simeq$	$Cl^+(0,3,1)$
$\mathbb{H}$	$1, i, j, k$	$\leftrightarrow$	$1,\ \mathbf{e}_2\mathbf{e}_3,\ \mathbf{e}_3\mathbf{e}_1,\ \mathbf{e}_1\mathbf{e}_2$
$\varepsilon\mathbb{H}$	$\varepsilon, i\varepsilon, j\varepsilon, k\varepsilon$	$\leftrightarrow$	$-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4,\ \mathbf{e}_1\mathbf{e}_4,\ \mathbf{e}_2\mathbf{e}_4,\ \mathbf{e}_3\mathbf{e}_4$

1.2 Clifford algebra $Cl(0,3,1)$

1. **Blades** of $Cl(0,3,1)$ express geometric elements of $\mathbb{R}^3$:
   - **1-blade**
     $\pi = n_x\mathbf{e}_1 + n_y\mathbf{e}_2 + n_z\mathbf{e}_3 + d\mathbf{e}_4 \quad \longleftrightarrow$ **Plane**: $\boldsymbol{n} \cdot \boldsymbol{x} = d$
   - **2-blade**
     $\ell = (v_0^x\mathbf{e}_2\mathbf{e}_3 + v_0^y\mathbf{e}_3\mathbf{e}_1 + v_0^z\mathbf{e}_1\mathbf{e}_2) + (v_1^x\mathbf{e}_1\mathbf{e}_4 + v_1^y\mathbf{e}_2\mathbf{e}_4 + v_1^z\mathbf{e}_3\mathbf{e}_4)$
     with $|\boldsymbol{v}_0| = 1$, $\boldsymbol{v}_0 \cdot \boldsymbol{v}_1 = 0 \quad \longleftrightarrow$ **Line**: $\boldsymbol{x} = \boldsymbol{v}_0 \times \boldsymbol{v}_1 + t\boldsymbol{v}_0$ $(t \in \mathbb{R})$
   - **3-blade**
     $p = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + x\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 + y\mathbf{e}_3\mathbf{e}_1\mathbf{e}_4 + z\mathbf{e}_1\mathbf{e}_2\mathbf{e}_4 \longleftrightarrow$ **Point**: $\boldsymbol{x} = (x, y, z)$

2. Algebraic operations in $Cl(0,3,1)$ (up to real positive multiples) express geometric manipulations:
   - **exterior product**: $\ell = \pi_1 \wedge \pi_2, \quad p = \ell \wedge \pi, \quad p = \pi_1 \wedge \pi_2 \wedge \pi_3$
     e.g., the intersection line ℓ of two planes π_1 and π_2 is expressed by the 2-blade $\pi_1 \wedge \pi_2$; the 2-blade is zero if and only if the two planes are parallel.
   - **Shuffle product**: $\ell = p_1 \vee p_2, \quad \pi = p \vee \ell, \quad \pi = p_1 \vee p_2 \vee p_3$
     e.g., the product of two points p_1, p_2 expresses the line ℓ passing through both points.
   - **Contraction**: $\pi^\perp = \pi \rfloor \ell$
     e.g., the 1-blade $\pi \rfloor \ell$ expresses the plane which contains ℓ and is perpendicular to π.
   - **Euclidean motions**: $Sp(1) \ltimes \mathbb{R}^3 \subset \mathbb{H} \oplus \varepsilon\mathbb{H} = Cl^+(0,3,1)$ is a double cover of the group of Euclidean motions $SE(3) = SO(3) \ltimes \mathbb{R}^3$ so that $\pm\check{q} \in Sp(1) \ltimes \mathbb{R}^3$ defines an Euclidean motions

$$\Theta(\check{q}) : \mathbb{R}^3 \to \mathbb{R}^3 \ \text{ s.t } \ 1 + \varepsilon\Theta(\check{q})(\boldsymbol{x}) = \check{q}(1 + \varepsilon\boldsymbol{x})\check{q}^*.$$

3. **Dual vectors** $\mathbb{D}^3 := \{\check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 \mid \boldsymbol{v}_0, \boldsymbol{v}_1 \in \mathbb{R}^3\}$ is the Lie algebra of $Sp(1) \ltimes \mathbb{R}^3$:

$$T_e(Sp(1) \ltimes \mathbb{R}^3) = T_e S^3 \oplus \varepsilon(\operatorname{Im}\mathbb{H}) = (\operatorname{Im}\mathbb{H}) \oplus \varepsilon(\operatorname{Im}\mathbb{H}) = \mathbb{D}^3.$$

- Inner product and exterior product (Lie bracket) on $\mathbb{D}^3$:

$$\check{\boldsymbol{u}} \cdot \check{\boldsymbol{v}} := -\tfrac{1}{2}(\check{\boldsymbol{u}}\check{\boldsymbol{v}} + \check{\boldsymbol{v}}\check{\boldsymbol{u}}) \in \mathbb{D}, \quad \check{\boldsymbol{u}} \times \check{\boldsymbol{v}} := \tfrac{1}{2}(\check{\boldsymbol{u}}\check{\boldsymbol{v}} - \check{\boldsymbol{v}}\check{\boldsymbol{u}}) = \tfrac{1}{2}[\check{\boldsymbol{u}}, \check{\boldsymbol{v}}] \in \mathbb{D}^3.$$

- **Dual vectors** = 2-blade ℓ:
  $\check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 \in \mathbb{D}^3$ with $|\boldsymbol{v}_0| = 1$, $\boldsymbol{v}_0 \cdot \boldsymbol{v}_1 = 0 \longleftrightarrow$ **oriented line** in $\mathbb{R}^3$
  - $\boldsymbol{a} \in \mathbb{R}^3$ lies on $L_{\check{\boldsymbol{v}}} \iff \boldsymbol{a} \times \boldsymbol{v}_0 = \boldsymbol{v}_1$;
  - $L_{\check{\boldsymbol{u}}}$ and $L_{\check{\boldsymbol{v}}}$ intersect perpendicularly $\iff \check{\boldsymbol{u}} \cdot \check{\boldsymbol{v}} = 0$.

2 Classical line geometry

2.1 Dual Frenet formula

A **ruled surface** is described as a curve

$$\check{\boldsymbol{v}} : I \to \mathbb{D}^3, \quad \check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s)$$

with $|\boldsymbol{v}_0(s)| = 1$ and $\boldsymbol{v}_0(s) \cdot \boldsymbol{v}_1(s) = 0$ (I an open interval). It gives a canonical parametrization

$$F : I \times \mathbb{R} \to \mathbb{R}^3, \quad F(s,t) = \boldsymbol{r}(s) + t\boldsymbol{e}(s)$$

($\boldsymbol{r} = \boldsymbol{v}_0 \times \boldsymbol{v}_1$, $\boldsymbol{e} = \boldsymbol{v}_0$). That leads us to define the **dual curvature** by

$$\check{\kappa}(s) = \kappa_0(s) + \varepsilon\kappa_1(s) := \sqrt{\check{\boldsymbol{v}}'(s) \cdot \check{\boldsymbol{v}}'(s)} = |\boldsymbol{v}_0'| + \varepsilon \frac{\boldsymbol{v}_0' \cdot \boldsymbol{v}_1'}{|\boldsymbol{v}_0'|} \in \mathbb{D},$$

provided $\check{\boldsymbol{v}}$ is non-cylindrical, i.e., $\boldsymbol{v}_0'(s) \neq 0$ ($s \in I$). Here $'$ means $\frac{d}{ds}$. Note that $\check{\kappa}$ is invertible in $\mathbb{D}$.

From now on, we assume that s is the arc-length of $\boldsymbol{v}_0$; $\kappa_0(s) = |\boldsymbol{v}_0'(s)| = 1$. Put

$$\check{\boldsymbol{n}}(s) = \boldsymbol{n}_0(s) + \varepsilon \boldsymbol{n}_1(s) := \check{\kappa}^{-1}\check{\boldsymbol{v}}'(s), \quad \check{\boldsymbol{t}}(s) = \boldsymbol{t}_0(s) + \varepsilon \boldsymbol{t}_1(s) := \check{\boldsymbol{v}}(s) \times \check{\boldsymbol{n}}(s).$$

Then $\check{\boldsymbol{v}}(s)$, $\check{\boldsymbol{n}}(s)$ and $\check{\boldsymbol{t}}(s)$ form a basis of $\mathbb{D}^3$ (as a module over $\mathbb{D}$) which satisfy

$$\check{\boldsymbol{v}} \times \check{\boldsymbol{n}} = \check{\boldsymbol{t}}, \quad \check{\boldsymbol{t}} \times \check{\boldsymbol{v}} = \check{\boldsymbol{n}}, \quad \check{\boldsymbol{n}} \times \check{\boldsymbol{t}} = \check{\boldsymbol{v}},$$

$$\check{\boldsymbol{v}} \cdot \check{\boldsymbol{n}} = \check{\boldsymbol{n}} \cdot \check{\boldsymbol{t}} = \check{\boldsymbol{t}} \cdot \check{\boldsymbol{v}} = 0, \quad \check{\boldsymbol{v}} \cdot \check{\boldsymbol{v}} = \check{\boldsymbol{n}} \cdot \check{\boldsymbol{n}} = \check{\boldsymbol{t}} \cdot \check{\boldsymbol{t}} = 1.$$

The **dual torsion** $\check{\tau}(s)$ of $\check{\boldsymbol{v}}$ is defined by

$$\check{\tau}(s) = \tau_0(s) + \varepsilon\tau_1(s) := \check{\boldsymbol{n}}'(s) \cdot \check{\boldsymbol{t}}(s) \quad \in \mathbb{D}.$$

Theorem 2.1 (cf. Guggenheimmer [1, §8.2], Hlavatý [2]) Assume that $\check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 : I \to \mathbb{D}^3$ with the parameter s being the arc-length of $\boldsymbol{v}_0$, i.e., $\kappa_0 = 1$.

1. (Frenet formula) It holds that

$$\frac{d}{ds}\begin{bmatrix} \check{\boldsymbol{v}}(s) \\ \check{\boldsymbol{n}}(s) \\ \check{\boldsymbol{t}}(s) \end{bmatrix} = \begin{bmatrix} 0 & \check{\kappa}(s) & 0 \\ -\check{\kappa}(s) & 0 & \check{\tau}(s) \\ 0 & -\check{\tau}(s) & 0 \end{bmatrix}\begin{bmatrix} \check{\boldsymbol{v}}(s) \\ \check{\boldsymbol{n}}(s) \\ \check{\boldsymbol{t}}(s) \end{bmatrix}$$

2. (Completeness) Two possibly singular ruled surfaces in $\mathbb{R}^3$ are transformed to each other by some Euclidean motion if and only if their dual curvatures and dual torsions $\check{\kappa}$, $\check{\tau}$ coincide, i.e., κ_1, τ_0, τ_1 are complete invariants of a non-cylindrical ruled surface.
3. (Developable) Guassian curvature =0 if and only if $\kappa_1 = 0$ identically. In particular, τ_0, τ_1 are complete invariants of a non-cylindrical developable surface.

2.2 Dual Bouquet formula

For every $s \in I$, three lines in $\mathbb{R}^3$ corresponding to unit dual vectors $\check{\boldsymbol{v}}(s), \check{\boldsymbol{n}}(s), \check{\boldsymbol{t}}(s)$ are mutually perpendicular and meet at one point, say $\sigma(s)$, which is known as a striction point. So, in $\mathbb{R}^3$, direction vectors $\boldsymbol{v}_0(s), \boldsymbol{n}_0(s), \boldsymbol{t}_0(s)$ form a **moving frame** along the **striction curve** $\sigma(s)$. By an Euclidean motion, we may assume that

$$\check{\boldsymbol{v}}(0) = [1,0,0]^T,\ \check{\boldsymbol{n}}(0) = [0,1,0]^T,\ \check{\boldsymbol{t}}(0) = [0,0,1]^T \quad \in \mathbb{D}^3,$$

that is, $\{\boldsymbol{v}_0(0), \boldsymbol{n}_0(0), \boldsymbol{t}_0(0)\}$ is the standard basis and $\sigma(0) = 0$ ($\Leftrightarrow$ $\boldsymbol{v}_1(0) = \boldsymbol{n}_0(0) = \boldsymbol{t}_0(0) = 0$).

By iterating the Frenet formula, we obtain the **Bouquet formula** at $s = 0$;

$$\check{\boldsymbol{v}}(s) = \sum_{n=0}^{r} \frac{\check{\boldsymbol{v}}^{(n)}(0)}{n!} s^n + o(r) = \begin{bmatrix} 1 - \frac{1}{2}\check{\kappa}^2 s + \cdots \\ \check{\kappa} s + \frac{1}{2}\check{\kappa}' s + \cdots \\ \frac{1}{2}\check{\kappa}\check{\tau} s^2 + \cdots \end{bmatrix} \quad \in \mathbb{D}^3.$$

Convention : $\check{\kappa}, \check{\tau}, \check{\kappa}', \check{\tau}', \cdots$ denote their values at $s = 0$, e.g. $\check{\kappa}' = \check{\kappa}'(0)$, unless specifically mentioned.

Substitute $\check{\kappa} = \kappa_0 + \varepsilon\kappa_1$ and $\check{\tau} = \tau_0 + \varepsilon\tau_1$, we get the Taylor expansion of the map $F(s,t) = \boldsymbol{v}_0(s) \times \boldsymbol{v}_1(s) + t\boldsymbol{v}_0(s)$ at a point $(0, t_0)$ lying on the ruling of $s = 0$. It turns out that F is singular at $(0, t_0)$ iff $t_0 = 0$ (i.e. striction point) and $\kappa_1(0) = 0$. Then F is expanded at $(0,0)$ as

$$\left\{ \begin{array}{lcl} x & = & t - \frac{1}{2}ts^2 + \frac{\tau_1}{2}s^3 + \cdots \\ y & = & ts - \frac{\tau_1}{2}s^2 - \frac{2\tau_0\kappa_1' + \tau_1'}{6}s^3 + \cdots \\ z & = & \frac{\kappa_1'}{2}s^2 + \frac{\tau_0}{2}ts^2 + \frac{\kappa_1'' - 2\tau_0\tau_1}{6}s^3 + \cdots \end{array} \right. \qquad (*)$$

3 Singularities of Ruled and Developable Surfaces

3.1 Equivalences

Let $f, g : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ be map-germs.

- f and g are **$\mathcal{A}$-equivalent** if $\exists\,(\sigma, \tau) \in \mathcal{A} := \mathrm{Diff}(\mathbb{R}^m, 0) \times \mathrm{Diff}(\mathbb{R}^n, 0)$ s.t. $g = \tau \circ f \circ \sigma^{-1}$.
- **Rigid equivalence** (tentatively): up to $(\sigma, \tau) \in \mathrm{Diff}(\mathbb{R}^m, 0) \times SO(n)$.

Of our interest is to classify the germs of parametrizations $F : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ of ruled surfaces up to $\mathcal{A}$-equivalence and rigid equivalence.

3.2 $\mathcal{A}$-recognition of ruled surfaces

Crosscap $S_0 : (x, xy, y^2)$ is 2-$\mathcal{A}$-determined. Hence, by the above expansion (*) of F, we see that

$$F \sim_{\mathcal{A}} S_0 \iff \kappa_1(0) = 0,\ \kappa_1'(0) \neq 0$$

In case of $\kappa_1(0) = \kappa_1'(0) = 0$, $j^2F(0) \sim_{\mathcal{A}} (x, y^2, 0)$ or $(x, xy, 0)$ according to whether $\tau_1(0) \neq 0$ or $= 0$. Then, applying Mond's $\mathcal{A}$-recognition tree [10], we obtain

Theorem 3.1 [13] For a non-cylindrical ruled surface ($\kappa_0 = 1$),

1. there is a unique singular point on the ruling $L_{\check{\boldsymbol{v}}(s_0)}$ iff $\kappa_1(s_0) = 0$;
2. **$\mathcal{A}$-classification** of singularities of F arising in generic at most 3-parameter families of non-cylindrical ruled surfaces is given in Table 1;
3. For each $\mathcal{A}$-type, κ_1, τ_0, τ_1 with the condition gives a normal form of the ruled surface-germ in **rigid classification** by solving the Frenet ODE; its jet is given by (*).

Remark 3.2

1. The generic case (i.e. crosscap S_0) was firstly proved in Izumiya-Takeuchi [6] in a rigorous way. Martins and Nuño-Ballesteros [9] showed that any $\mathcal{A}$-simple map-germ is equivalent to a germ of non-cylindrical ruled surface.
2. From our theorem, all $\mathcal{A}$-types of codim ≤ 5 are realized by ruled surface-germs. Indeed, there is an $\mathcal{A}$-type of codim 6 which is not realized, e.g., the 3-jet (x, y^3, x^2y) and the 5-jet (x, y^2, x^4y) is not equivalent to jets of any (cylindrical/non-cylindrical) ruled surfaces.
3. For each type, $\mathcal{A}_e$-versal deformation is realized via deforming κ_1, τ_0, τ_1 properly.

	normal form	ℓ	cond. at $s = s_0$
S_0	(x, y^2, xy)	2	$\kappa_1 = 0,\ \kappa_1' \neq 0$
$S_1^{\pm}$	$(x, y^2, y^3 \pm x^2 y)$	3	$\kappa_1 = \kappa_1' = 0,\ \tau_1 \neq 0,\ \kappa_1''(\kappa_1'' - 2\tau_0\tau_1) \gtrless 0$
S_2	$(x, y^2, y^3 + x^3 y)$	4	$\kappa_1 = \kappa_1' = \kappa_1'' = 0,\ \kappa_1^{(3)}\tau_0\tau_1 \neq 0$
$B_2^{\pm}$	$(x, y^2, x^2 y \pm y^5)$		$\kappa_1 = \kappa_1' = 0,\ \kappa_1'' = 2\tau_0\tau_1 \neq 0,\ b_2 \gtrless 0$
H_2	$(x, xy + y^5, y^3)$		$\kappa_1 = \kappa_1' = \tau_1 = 0,\ \kappa_1'' \neq 0,\ h_2 \neq 0$
$S_3^{\pm}$	$(x, y^2, y^3 \pm x^4 y)$	5	$\kappa_1 = \kappa_1' = \kappa_1'' = \kappa_1^{(3)} = 0,\ \kappa_1^{(4)}\tau_0\tau_1 \gtrless 0$
$C_3^{\pm}$	$(x, y^2, xy^3 \pm x^3 y)$		$\kappa_1 = \kappa_1' = \kappa_1'' = \tau_0 = 0,\ \tau_1 \neq 0,\ \kappa_1^{(3)}(\kappa_1^{(3)} - 2\tau_0'\tau_1) \gtrless 0$
$B_3^{\pm}$	$(x, y^2, x^2 y \pm y^7)$		$\kappa_1 = \kappa_1' = 0,\ \kappa_1'' = 2\tau_0\tau_1 \neq 0,\ b_2 = 0,\ b_3 \gtrless 0$
H_3	$(x, xy + y^7, y^3)$		$\kappa_1 = \kappa_1' = \tau_1 = 0,\ \kappa_1'' \neq 0,\ h_2 = 0,\ h_3 \neq 0$
P_3	$(x, xy + y^3, xy^2 + p_4 y^4)$		$\kappa_1 = \kappa_1' = \kappa_1'' = \tau_1 = 0,\ \tau_0\tau_1' \neq 0,\ p_4 \neq 0, 1, \frac{1}{2}, \frac{3}{2}.$

Table 1 Characterization of germs of ruled surfaces. There are certain polynomials b_2, b_3, h_2, h_3, p_4 in derivatives of κ_1, τ_0, τ_1 [13]. ℓ is $\mathcal{A}$-codimension of the germ.

3.3 $\mathcal{A}$-recognition of developable surfaces

Theorem 3.3 [13] For a non-cylindrical developable surface ($\kappa_0 = 1, \kappa_1 = 0$),

1. It is the tangent developable of the striction curve given by $\sigma(s) := F(s, -\boldsymbol{r}'(s) \cdot \boldsymbol{e}'(s))$ $(\boldsymbol{r} = \boldsymbol{v}_0 \times \boldsymbol{v}_1, \boldsymbol{e} = \boldsymbol{v}_0)$;
2. **$\mathcal{A}$-classification** of singularities of F arising in generic at most 2-parameter families of non-cylindrical developable surfaces is given in Table 2;
3. For each $\mathcal{A}$-type, τ_0, τ_1 with the condition gives a normal form of the developable surface-germ in **rigid classification** by solving the Frenet ODE; its jet is given by (*).

Remark 3.4 (i) Izumiya-Takeuchi [6] classified generic singularities of developable surfaces rigorously, and Kurokawa [8] treated 1-parameter families of developables. Our result generalizes those.
(ii) Some $\mathcal{A}$-types of frontal-germs are not realized by non-cylindrical developables.

- cS_1^- : $(x, y^2, y^3(x^2 - y^2))$ and cC_3^- : $(x, y^2, y^3(x^3 - xy^2))$ never appear.
- $\tau_1 \neq 0$ and $\tau_0 = \tau_0' = \tau_0'' = 0$ iff $j^5 F \sim_A (x, y^2, 0)$. Thus, cS : $(x, y^2, y^3(y^2 + h(x, y^2)))$ and cB : $(x, y^2, y^3(x^2 + h(x, y^2)))$ with $h(x, y^2) = o(2)$ never appear.
- $\tau_1 = 0$ iff $j^2 F \sim_A (x, xy, 0)$. Thus cuspidal beaks/lips type $A_3^{\pm}$ and purse/pyramid types D_k never appear (indeed, their 2-jets are equivalent to $(x, 0, 0)$ and $(x^2 \pm y^2, xy, 0)$ respectively).

	normal form	ℓ	cond. at $s = s_0$
cE	(x, y^2, y^3)	1	$\tau_0 \neq 0,\ \ \tau_1 \neq 0$
cS_0	(x, y^2, xy^3)	2	$\tau_1 \neq 0,\ \ \tau_0 = 0,\ \ \tau_0' \neq 0$
cS_1^+	$(x, y^2, y^3(x^2 + y^2))$	3	$\tau_1 \neq 0,\ \ \tau_0 = \tau_0' = 0,\ \ \tau_0'' \neq 0$
cC_3^+	$(x, y^2, y^3(x^3 + xy^2))$	4	$\tau_1 \neq 0,\ \ \tau_0 = \tau_0' = \tau_0'' = 0,\ \ \tau_0''' \neq 0$
Sw	$(x, xy + 2y^3, xy^2 + 3y^4)$	2	$\tau_0 \neq 0,\ \ \tau_1 = 0,\ \ \tau_1' \neq 0$
cA_4	$(x, xy + \frac{5}{2}y^4, xy^2 + 4y^5)$	3	$\tau_0 \neq 0,\ \ \tau_1 = \tau_1' = 0,\ \ \tau_1'' \neq 0$
cA_5	$(x, xy + 3y^5, xy^2 + 5y^6)^\dagger$	4	$\tau_0 \neq 0,\ \ \tau_1 = \tau_1' = \tau_1'' = 0,\ \ \tau_1''' \neq 0$
T_1	$(x, xy + y^3, 0) + o(3)$	3	$\tau_0 = \tau_1 = 0,\ \ \tau_1' \neq 0$
T_2	$(x, xy, 0) + o(3)$	4	$\tau_0 = \tau_1 = \tau_1' = 0$

Table 2 Characterization of germs of developable surfaces. †: topological $\mathcal{A}$-equivalence.

A space curve-germ is called to be of type $(m, m+\ell, m+\ell+r)$ if it is $\mathcal{A}$-equivalent to the germ

$$x = s^m + o(m), \quad y = s^{m+\ell} + o(m+\ell), \quad z = s^{m+\ell+r} + o(m+\ell+r)$$

Theorem 3.5 (G. Ishikawa [4]) **Topological type** of the tangent developable of a space curve is uniquely determined by type $(m, m+\ell, m+\ell+r)$ of the curve, unless both ℓ, r are even.

Theorem 3.6 (Topological classification [13]) For a non-cylindrical developable surface, the germ of its striction curve $\sigma(s)$ at $s = s_0$ has the type $(m,\ m+1,\ m+1+r)$, if the orders at $s = s_0$ are: $\tau_1(s) = o(m-2)$ and $\tau_0(s) = o(r-2)$. In particular, the **topological type of** F at a singular point is uniquely determined by vanishing orders of the **dual torsion** $\check{\tau} = \tau_0 + \varepsilon\tau_1$.

This generalizes a known result that the $\mathcal{A}$-type of the tangent developable of a non-singular space curve σ with non-zero curvature is uniquely determined by the vanishing order of its torsion function (Ishikawa [4]); that is the case of $(1, 2, 2+r)$ (i.e., $\tau_1(s_0) \neq 0$) and then the order of τ_0 is equal to the order of torsion of σ.

4 Further discussion

4.1 Line congruence and line complex

Consider a 2-parameter family of lines $\check{\boldsymbol{v}} : U \to \mathbb{D}^3$, $\check{\boldsymbol{v}}(p) = \boldsymbol{v}_0(p) + \varepsilon\boldsymbol{v}_1(p)$, with $|\boldsymbol{v}_0| = 1$ and $\boldsymbol{v}_0 \cdot \boldsymbol{v}_1 = 0$, $U \subset \mathbb{R}^2$ an open subset, which defines a *line congruence*. It is parameterized by the map $F : U \times \mathbb{R} \to \mathbb{R}^3$, $F(p,t) = \boldsymbol{v}_0(p) \times \boldsymbol{v}_1(p) + t\boldsymbol{v}_0(p)$. The

Frenet formula for the Dorboux frame in $\mathbb{D}^3$ is available: $\exists\, \omega_i \in \Omega^1(U)$ s.t.

$$d \begin{bmatrix} \check{\boldsymbol{v}}(p) \\ \check{\boldsymbol{n}}(p) \\ \check{\boldsymbol{t}}(p) \end{bmatrix} = \begin{bmatrix} 0 & \omega_1(p) & \omega_2(p) \\ -\omega_1(p) & 0 & \omega_3(p) \\ -\omega_2(p) & -\omega_3(p) & 0 \end{bmatrix} \begin{bmatrix} \check{\boldsymbol{v}}(p) \\ \check{\boldsymbol{n}}(p) \\ \check{\boldsymbol{t}}(p) \end{bmatrix}$$

(cf. Guggenheimmer [1, §10]). This kind of Frenet formula is also available for a family of lines with 3 or more parameters, called a *line complex*. We can obtain $\mathcal{A}$-classification of singularities of line congruences and line complexes by using

- $\mathcal{A}$-classification of $\mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ (Bruce, Marar-Tari, Hawes)
- $\mathcal{A}$-classification of $\mathbb{R}^4, 0 \to \mathbb{R}^3, 0$ (A. C. Nabarro)

4.2 Other Clifford Algebra

- Higher dimensional case $Cl^+(0, n, 1) \Longrightarrow$ Ruled objects in $\mathbb{R}^n$.
- **Conformal Geometric Algebra** $\simeq Cl(4, 1, 0) \Longrightarrow$ envelopes of circles, shperes, etc. e.g. Sing. of families of horospheres, etc. (Izumiya-Saji-Takahashi [5])
- Projectivized Clifford Algebra $\Longrightarrow$ projective differential geometry (Wilczynski, Kabata [7])

4.3 Curves and surfaces in $\mathbb{D}^3$

A curve of dual vectors, $I \to \mathbb{D}^3$, is called a *framed curve*, which describes a 1-parameter family of Euclidean motions of $\mathbb{R}^3$. There is also a Frenet-type formula and various aspects of singular objects associated to framed curves have been studied by Honda-Takahashi [3]. It would be interesting to reformulate the theory of frontal surfaces in $\mathbb{R}^3$ as surface theory in $\mathbb{D}^3$.

4.4 Hybrid approach with discrete differential geometry

How to discretize ruled/developable surfaces around singular points ? As seen above, we have obtained rigid classification of singularities of ruled/developable surfaces; the curve-germs in $\mathbb{D}^3$ is determined by jets of $\check{\kappa}, \check{\tau}$. Therefore, we may first discretize the curves in $\mathbb{D}^3$ with respect to the parameter s and then discretize rulings with respect to the parameter t. Semi-algebraic (e.g. Bézier) versions can also be considered. This approach might be interesting for singularity analysis in several applications from pure math. to applied math. ; (classical) integrable systems, architectural geometry, data analysis (surface fittings), computer visions, robotics and so on.

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